Online Appendix
Scaled PCA: A New Approach to Dimension Reduction

A Theoretical proof for the case of weak factors

To prove Propositions 1 and 2, we need the following lemmas.

Lemma 1. *Let $\widetilde{V}$ be the diagonal matrix whose diagonal elements are the largest two eigenvalues of the matrix $\sum_{i=1}^{N} Z_i Z_i'$ with $Z_i = \dot{F}\lambda_i\widehat{\gamma}_i + \widehat{\gamma}_i\dot{e}_i$. Under Assumptions 1-5, together with $\frac{N^{1-\nu}}{T^2} \to 0$, we have*

$$\widetilde{V} \asymp_p N^{\nu} T.$$

Proof of Lemma 1. Note that $\widetilde{V}$ is robust to the rotational indeterminacy. So it is no loss of generality to impose the normalization condition $\frac{1}{T}\dot{F}'\dot{F} = I$. Because $\frac{1}{T}\sum_{t=1}^{T}\dot{X}_{it}^2 = 1$, it is easy to see that

$$\widehat{\gamma}_i = \frac{1}{T}\sum_{t=1}^{T}\dot{X}_{it}\dot{y}_{t+h} = \frac{1}{T}\sum_{t=1}^{T}(\phi_i\dot{g}_t + \psi_i\dot{h}_t + \dot{e}_{it})(\beta\dot{g}_t + \dot{\epsilon}_{t+h}) \tag{A1}$$

$$= \phi_i\beta + \beta\frac{1}{T}\sum_{t=1}^{T}e_{it}\dot{g}_t + \phi_i\frac{1}{T}\sum_{t=1}^{T}\dot{g}_t\dot{\epsilon}_{t+h} + \psi_i\frac{1}{T}\sum_{t=1}^{T}\dot{h}_t\dot{\epsilon}_{t+h} + \frac{1}{T}\sum_{t=1}^{T}\dot{e}_{it}\dot{\epsilon}_{t+h} \equiv \gamma_i + \mathbf{u}_i \tag{A2}$$

with $\gamma_i = \beta\phi_i$ and

$$\mathbf{u}_i = \underbrace{\frac{1}{T}\sum_{t=1}^{T}e_{it}(\beta\dot{g}_t + \dot{\epsilon}_{t+h})}_{\mathbf{u}_{i,1}} + \underbrace{\phi_i\frac{1}{T}\sum_{t=1}^{T}\dot{g}_t\dot{\epsilon}_{t+h} + \psi_i\frac{1}{T}\sum_{t=1}^{T}\dot{h}_t\dot{\epsilon}_{t+h}}_{\mathbf{u}_{i,2}} \equiv \mathbf{u}_{i,1} + \mathbf{u}_{i,2}.$$

Let $\widetilde{F}$ and $\widetilde{V}$ be the eigenvectors and eigenvalues of the matrix $\sum_{i=1}^{N} Z_i Z_i$, we therefore have

$$\widetilde{V} = \widetilde{F}'\left[\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\lambda_i'\dot{F}' + \dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\dot{e}_i' + \sum_{i=1}^{N}\widehat{\gamma}_i^2\dot{e}_i\lambda_i'\dot{F}' + \sum_{i=1}^{N}\widehat{\gamma}_i^2\dot{e}_i\dot{e}_i'\right]\widetilde{F} = I_1 + \cdots + I_4, \quad \text{say.}$$

We will show that

$$\widetilde{V} = O_p(N^{\nu}T) + O_p(\frac{N}{T}) + O_p(\sqrt{N}). \tag{A3}$$

Let $\mathcal{I}_\phi$ be the set of units whose ϕ_i is zero and $\mathcal{I}_\phi^c$ be its complement set. We analyze the above four terms one by one. Consider I_1. Note that $\widehat{\gamma}_i = \gamma_i + \mathbf{u}_i$ in set $\mathcal{I}_\phi$, and $\widehat{\gamma}_i = \mathbf{u}_i$ in set $\mathcal{I}_\phi^c$. Thus,

$$\begin{aligned} I_1 &= \widetilde{F}'\dot{F}\Big[\sum_{i\in\mathcal{I}_\phi}\widehat{\gamma}_i^2\lambda_i\lambda_i' + \sum_{i\notin\mathcal{I}_\phi}\widehat{\gamma}_i^2\lambda_i\lambda_i'\Big]\dot{F}'\widetilde{F} \\ &= \widetilde{F}'\dot{F}\Big[\sum_{i\in\mathcal{I}_\phi}(\gamma_i^2 + \mathbf{u}_i^2 + 2\gamma_i\mathbf{u}_i)\lambda_i\lambda_i' + \sum_{i\notin\mathcal{I}_\phi}\mathbf{u}_i^2\lambda_i\lambda_i'\Big]\dot{F}'\widetilde{F} \\ &= \widetilde{F}'\dot{F}\Big[\sum_{i\in\mathcal{I}_\phi}(\gamma_i^2 + 2\gamma_i\mathbf{u}_i)\lambda_i\lambda_i' + \sum_{i=1}^{N}\mathbf{u}_i^2\lambda_i\lambda_i'\Big]\dot{F}'\widetilde{F} \end{aligned}$$

It is easy to see that $\sum_{i\in\mathcal{I}_\phi}\gamma_i^2\lambda_i\lambda_i' \asymp_p N^\nu$ and

$$\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\lambda_i\lambda_i' = \sum_{i\in\mathcal{I}_\phi}\gamma_i(\mathbf{u}_{i,1} + \mathbf{u}_{i,2})\lambda_i\lambda_i' = \sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,1}\lambda_i\lambda_i' + O_p(N^\nu T^{-1/2}).$$

Let $q_s = \beta\dot{g}_s + \dot{\epsilon}_{s+h}$, also note that

$$\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,1}\lambda_i\lambda_i' = \frac{1}{T}\sum_{i\in\mathcal{I}_\phi}\sum_{s=1}^{T}e_{is}q_s\gamma_i\lambda_i\lambda_i' = O_p(N^{\nu/2}T^{-1/2}).$$

With the above two results, we therefore have

$$\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\lambda_i\lambda_i' = O_p(N^{\nu/2}T^{-1/2}) + O_p(N^\nu T^{-1/2}) = O_p(N^\nu T^{-1/2}).$$

Proceed to consider the term $\sum_{i=1}^N \mathbf{u}_i^2\lambda_i\lambda_i'$. By the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{N}\mathbf{u}_i^2\lambda_i\lambda_i' \leq 2\sum_{i=1}^{N}(\mathbf{u}_{i,1}^2 + \mathbf{u}_{i,2}^2)\lambda_i\lambda_i' = O_p(N^\nu T^{-1}).$$

Given the above result, we conclude

$$I_1 \asymp_p \widetilde{F}'\dot{F}\sum_{i=1}^{N}\gamma_i^2\lambda_i\lambda_i'\dot{F}\widetilde{F} \asymp_p N^\nu T. \tag{A4}$$

Consider I_2. Similarly, it can be written as

$$I_2 = \widetilde{F}'\dot{F}\Big[\sum_{i\in\mathcal{I}_\phi}(\gamma_i^2 + 2\gamma_i\mathbf{u}_i)\lambda_i\dot{e}_i' + \sum_{i=1}^{N}\mathbf{u}_i^2\lambda_i\dot{e}_i'\Big]\widetilde{F}.$$

As shown before, $\sum_{t=1}^{T}\widetilde{f}_t = 0$ and therefore $\dot{e}_i'\widetilde{F} = e_i'\widetilde{F}$. So

$$\Big\|\sum_{i\in\mathcal{I}_\phi}\gamma_i^2\lambda_i\dot{e}_i'\widetilde{F}\Big\| = \Big\|\sum_{i\in\mathcal{I}_\phi}\gamma_i^2\lambda_i e_i'\widetilde{F}\Big\| \le \Big[\sum_{t=1}^{T}\|\widetilde{f}_t\|^2\Big]^{1/2}\Big[\sum_{t=1}^{T}\Big\|\sum_{i\in\mathcal{I}_\phi}\gamma_i^2\lambda_i e_{it}\Big\|^2\Big]^{1/2} = O_p(N^{\nu/2}T^{1/2}).$$

Furthermore,

$$\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\lambda_i\dot{e}_i'\widetilde{F} = \sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,1}\lambda_i\dot{e}_i'\widetilde{F} + \sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,2}\lambda_i\dot{e}_i'\widetilde{F}.$$

By the same arguments, we have $\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,2}\lambda_i\dot{e}_i'\widetilde{F} = \sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,2}\lambda_i e_i'\widetilde{F} = O_p(N^{\nu/2})$. As regard to the first term,

$$\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,1}\lambda_i\dot{e}_i'\widetilde{F} = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}q_t\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i[e_{it}e_{is} - E(e_{it}e_{is})]\lambda_i\Big]\widetilde{f}_s$$
$$+ \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}q_t\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i E(e_{it}e_{is})\lambda_i\Big]\widetilde{f}_s = O_p(N^{\nu/2}) + O_p(N^{\nu}T^{-1/2}),$$

because

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}q_t\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i[e_{it}e_{is} - E(e_{it}e_{is})]\lambda_i\Big]\widetilde{f}_s \le \frac{1}{T}\Big[\sum_{s=1}^{T}\Big\|\sum_{t=1}^{T}\sum_{i\in\mathcal{I}_\phi}q_t\gamma_i[e_{it}e_{is} - E(e_{it}e_{is})]\lambda_i\Big\|^2\Big]^{1/2}\Big[\sum_{s=1}^{T}\|\widetilde{f}_s\|^2\Big]^{1/2},$$

where the right expression is $O_p(N^{\nu/2})$, and

$$\Big\|\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}q_t\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i E(e_{it}e_{is})\lambda_i\Big]\widetilde{f}_s\Big\| \le \frac{1}{T}\Big[\sum_{s=1}^{T}\Big\|\sum_{i\in\mathcal{I}_\phi}\sum_{t=1}^{T}q_t\gamma_i E(e_{it}e_{is})\lambda_i\Big\|^2\Big]^{1/2}\Big[\sum_{s=1}^{T}\|\widetilde{f}_s\|^2\Big]^{1/2} = O_p(N^{\nu}T^{-1/2}).$$

So we conclude that

$$\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\lambda_i\dot{e}_i'\widetilde{F} = O_p(N^{\nu/2}) + O_p(N^{\nu}T^{-1/2}).$$

Finally, we consider the last term in I_2.

$$\sum_{i=1}^{N}\mathbf{u}_i^2\lambda_i\dot{e}_i'\widetilde{F}=\sum_{i=1}^{N}(\mathbf{u}_{i,1}^2+2\mathbf{u}_{i,1}\mathbf{u}_{i,2}+\mathbf{u}_{i,2}^2)\lambda_i e_i'\widetilde{F}=O_p(N^{\nu}T^{-1/2}).$$

Given the above results, together with $\widetilde{F}'\dot{F}=O_p(\sqrt{T})$, we conclude that

$$I_2=O_p(N^{\nu/2}T)+O_p(N^{\nu}). \tag{A5}$$

Term I_3 is the transpose of I_2, so it has the same magnitude of I_2. Consider I_4.

$$I_4=\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}(\gamma_i^2+2\gamma_i\mathbf{u}_i)\dot{e}_i\dot{e}_i'\Big]\widetilde{F}+\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_i^2\dot{e}_i\dot{e}_i'\Big)\widetilde{F}=\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}(\gamma_i^2+2\gamma_i\mathbf{u}_i)e_ie_i'\Big]\widetilde{F}+\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_i^2e_ie_i'\Big)\widetilde{F}.$$

First note that $\widetilde{F}'\sum_{i\in\mathcal{I}_\phi}\gamma_i^2E(e_ie_i')\widetilde{F}=O_p(N^{\nu})$ and

$$\Big\|\widetilde{F}'\sum_{i\in\mathcal{I}_\phi}\gamma_i^2[e_ie_i'-E(e_ie_i')]\widetilde{F}\Big\|\le\Big[\sum_{t=1}^{T}\|\widetilde{f}_t\|^2\Big]\Big[\sum_{t=1}^{T}\sum_{s=1}^{T}\Big\|\sum_{i\in\mathcal{I}_\phi}\gamma_i^2[e_ie_i'-E(e_ie_i')]\Big\|^2\Big]^{1/2}=O_p(N^{\nu/2}T).$$

We therefore have

$$\widetilde{F}'[\sum_{i\in\mathcal{I}_\phi}\gamma_i^2\dot{e}_i\dot{e}_i']\widetilde{F}=O_p(N^{\nu})+O_p(N^{\nu/2}T).$$

In addition,

$$\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\dot{e}_i\dot{e}_i'\Big]\widetilde{F}=\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_ie_ie_i'\Big]\widetilde{F}=\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,1}e_ie_i'\Big]\widetilde{F}+\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_{i,2}e_ie_i'\Big]\widetilde{F}$$

The second expression is $O_p(N^{\nu}T^{-1/2})+O_p(N^{\nu/2}T^{-1/2})$ according to the above analysis. The first expression is $O_p(N^{\nu/2}T^{1/2})$. Given this, we have

$$\widetilde{F}'\Big[\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\dot{e}_i\dot{e}_i'\Big]\widetilde{F}=O_p(N^{\nu/2}T^{1/2})+O_p(N^{\nu}T^{-1/2}).$$

Furthermore,

$$\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_i^2e_ie_i'\Big)\widetilde{F}\le 2\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2e_ie_i'\Big)\widetilde{F}+2\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,2}^2e_ie_i'\Big)\widetilde{F}.$$

For the second term, by the triangle inequality,

$$\Big\|\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,2}^2 e_i e_i'\Big)\widetilde{F}\Big\| \le \Big[\Big\|\frac{1}{T}\sum_{t=1}^{T}\dot{g}_t\epsilon_{t+h}\Big\|^2\Big]\Big[\sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}\phi_i^2 e_{it}e_{is}\Big] + \Big[\Big\|\frac{1}{T}\sum_{t=1}^{T}\dot{h}_t\epsilon_{t+h}\Big\|^2\Big]\Big[\sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}\psi_i^2 e_{it}e_{is}\Big]$$

However, we have

$$\begin{aligned}\Big\|\sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}\phi_i^2 e_{it}e_{is}\Big\| &\le \Big\|\sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}\phi_i^2[e_{it}e_{is}-E(e_{it}e_{is})]\Big\| + \Big\|\sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}\phi_i^2 E(e_{it}e_{is})\Big\| \\ &\le \Big[\sum_{t=1}^{T}\|\widehat{f}_t\|^2\Big]\Big[\sum_{t=1}^{T}\sum_{s=1}^{T}\Big\|\sum_{i\in\mathcal{I}_\phi}\phi_i^2[e_{it}e_{is}-E(e_{it}e_{is})]\Big\|^2\Big]^{1/2} + O_p(N^\nu) \\ &= O_p(N^{\nu/2}T^{1/2}) + O_p(N^\nu).\end{aligned}$$

Similarly,

$$\sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}\psi_i^2 e_{it}e_{is} = O_p(N^{\nu/2}T^{1/2}) + O_p(N^\nu).$$

With the above results, we have

$$\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,2}^2 e_i e_i'\Big)\widetilde{F} = O_p(N^{\nu/2}T^{-1/2}) + O_p(N^\nu T^{-1}).$$

As regard to the first term. It can be decomposed as

$$\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2 e_i e_i'\Big)\widetilde{F} = \widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2[e_i e_i' - E(e_i e_i')]\Big)\widetilde{F} + \widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2 E(e_i e_i')\Big)\widetilde{F}.$$

Let $\zeta_{i,tt'} = e_{it}e_{it'} - E(e_{it}e_{it'})$. The first expression now is

$$\begin{aligned}\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2[e_i e_i' - E(e_i e_i')]\Big)\widetilde{F} &= \sum_{t=1}^{T}\sum_{t'=1}^{T}\widetilde{f}_t\widetilde{f}_{t'}'\sum_{i=1}^{N}\mathbf{u}_{i,1}^2\zeta_{i,tt'} \\ &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\widetilde{f}_t\widetilde{f}_{t'}'\sum_{i=1}^{N}\Big(\sum_{s=1}^{T}\sum_{s'=1}^{T}q_s q_{s'}e_{is}e_{is}'\Big)\zeta_{i,tt'} \\ &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\widetilde{f}_t\widetilde{f}_{t'}'\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{s'=1}^{T}q_s q_{s'}\Big[\zeta_{i,ss'}\zeta_{i,tt'} - E(\zeta_{i,ss'}\zeta_{i,tt'})\Big] \\ &\quad + \frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\widetilde{f}_t\widetilde{f}_{t'}'\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{s'=1}^{T}q_s q_{s'}E(\zeta_{i,ss'}\zeta_{i,tt'})\end{aligned}$$

$$+\frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\widetilde{f}_t\widetilde{f}_{t'}'\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{s'=1}^{T}q_sq_{s'}E(\zeta_{i,ss'})\zeta_{i,tt'}.$$

By assumption, the first term is bounded in norm by

$$\Big[\sum_{t=1}^{T}\|\widetilde{f}_t\|^2\Big]\Big[\frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\Big\|\frac{1}{T}\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{s'=1}^{T}q_sq_{s'}\Big[\zeta_{i,ss'}\zeta_{i,tt'}-E(\zeta_{i,ss'}\zeta_{i,tt'})\Big]\Big\|^2\Big]^{1/2}=O_p(\sqrt{N}).$$

The second term is $O_p(\frac{N}{T})$ by the independent assumption and the third term is bounded in norm by

$$\Big[\sum_{t=1}^{T}\|\widetilde{f}_t\|^2\Big]\Big[\frac{1}{T^2}\sum_{t=1}^{T}\sum_{t'=1}^{T}\Big\|\sum_{i=1}^{N}\Big(\frac{1}{T}\sum_{s=1}^{T}\sum_{s'=1}^{T}q_sq_{s'}E(\zeta_{i,ss'})\Big)\zeta_{i,tt'}\Big]\Big\|^2\Big]^{1/2}=O_p(\sqrt{N}).$$

With these, we have

$$\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2[e_ie_i'-E(e_ie_i')]\Big)\widetilde{F}=O_p(\sqrt{N})+O_p(\frac{N}{T}).$$

Similarly, we can show that

$$\widetilde{F}'\Big(\sum_{i=1}^{N}\mathbf{u}_{i,1}^2E(e_ie_i')\Big)\widetilde{F}=O_p(\sqrt{N})+O_p(\frac{N}{T}).$$

We therefore have

$$I_4=O_p(N^{\nu/2}T)+O_p(\frac{N}{T})+O_p(\sqrt{N}).\tag{A6}$$

With the results (A4), (A5) and (A6), we have (A3). Given (A3) and (A4), we have Lemma 1. This completes the proof. □

Lemma 2. *Let $\widehat{F}^{\text{sPCA}}$ be the estimator of factors in the sPCA. Under Assumptions 1-5, there exists a invertible matrix R_{sPCA} such that*

$$\frac{1}{\sqrt{T}}\|\widehat{F}^{\text{sPCA}}-\dot{F}R_{\text{sPCA}}'\|\asymp_p N^{-\nu/2}+T^{-1}+\frac{N^{1-\nu}}{T^2}.$$

Proof of Lemma 2. The previous lemma shows that we need to use $N^\nu T$ to normalize the data $\sum_{i=1}^N Z_iZ_i'$. In the remaining proof, we use $\widehat{F}$ and R to denote the estimated factor and the rotational matrix in the sPCA if no confusion arises, i.e., $\widehat{F} = \widehat{F}^{\text{sPCA}}$ and $R = R_{\text{sPCA}}$. Now we have

$$\left[\frac{1}{N^\nu T}\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\lambda_i'\dot{F}' + \frac{1}{N^\nu T}\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\dot{e}_i' + \frac{1}{N^\nu T}\sum_{i=1}^{N}\widehat{\gamma}_i^2\dot{e}_i\lambda_i'\dot{F}' + \frac{1}{N^\nu T}\sum_{i=1}^{N}\widehat{\gamma}_i^2\dot{e}_i\dot{e}_i'\right]\widehat{F} = \widehat{F}\widehat{V}.$$

Let $R = \frac{1}{N^\nu T}\widehat{V}^{-1}\hat{F}'F\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\lambda_i'$. With this, we have

$$\begin{aligned}\frac{\|\widehat{F} - \dot{F}R'\|}{\sqrt{T}} &= \frac{1}{N^\nu T^{3/2}}\left\|\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\dot{e}_i'\hat{F}\hat{V}^{-1}\right\| + \frac{1}{N^\nu T^{3/2}}\left\|\sum_{i=1}^{N}\widehat{\gamma}_i^2\dot{e}_i\lambda_i'\dot{F}'\hat{F}\hat{V}^{-1}\right\| \\ &+ \frac{1}{N^\nu T^{3/2}}\left\|\sum_{i=1}^{N}\widehat{\gamma}_i^2\dot{e}_i\dot{e}_i'\hat{F}\hat{V}^{-1}\right\| = II_1 + II_2 + II_3. \end{aligned} \tag{A7}$$

However, the previous analysis has shown that $\widehat{V} - N^{-\alpha}\Lambda'\Lambda = o_p(1)$. Now we investigate the above three terms one by one. Consider II_1.

$$\begin{aligned}\frac{1}{N^\nu T^{3/2}}\left\|\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i\dot{e}_i'\hat{F}\hat{V}^{-1}\right\| &= \frac{1}{N^\nu T^{3/2}}\left\|\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'\hat{F}\hat{V}^{-1}\right\| \\ &\leq \frac{1}{N^\nu T^{3/2}}\left\|\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'(\hat{F} - \dot{F}R')\hat{V}^{-1}\right\| + \frac{1}{N^\nu T^{3/2}}\left\|\dot{F}\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'\dot{F}R'\hat{V}^{-1}\right\| \\ &\leq \frac{\|\dot{F}\|}{\sqrt{T}}\frac{\|\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'\|}{N^\nu\sqrt{T}}\frac{\|\widehat{F} - \dot{F}R'\|}{\sqrt{T}}\|\widehat{V}^{-1}\| + \frac{\|\dot{F}\|}{\sqrt{T}}\frac{\|\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'\dot{F}\|}{N^\nu T}\|R\|\|\widehat{V}^{-1}\|.\end{aligned}$$

For the first expression, by $\widehat{\gamma}_i = \gamma_i + \mathbf{u}_i$, we have

$$\left\|\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'\right\| = \left\|\sum_{i\in\mathcal{I}_\phi}\gamma_i^2\lambda_i e_i'\right\| + \left\|\sum_{i\in\mathcal{I}_\phi\cup\mathcal{I}_\psi}(2\gamma_i\mathbf{u}_i + \mathbf{u}_i^2)\lambda_i e_i\right\| = O_p(N^{\nu/2}T^{1/2}) + O_p(N^\nu).$$

So the first expression is of smaller order relative to $\frac{1}{\sqrt{T}}\|\widehat{F} - \dot{F}R'\|$, the expression of the left hand side of (A7). So it is negligible. Consider the second expression. Note that

$$\sum_{i=1}^{N}\widehat{\gamma}_i^2\lambda_i e_i'\dot{F} = \sum_{i=1}^{N}(\gamma_i^2 + 2\gamma_i\mathbf{u}_i + \mathbf{u}_i^2)\lambda_i e_i'\dot{F} = \sum_{i\in\mathcal{I}_\phi}\gamma_i^2\lambda_i e_i'\dot{F} + 2\sum_{i\in\mathcal{I}_\phi}\gamma_i\mathbf{u}_i\lambda_i e_i'\dot{F} + \sum_{i=1}^{N}\mathbf{u}_i^2\lambda_i e_i'\dot{F}$$

The first term on right hand side is $O_p(N^{\nu/2}T^{1/2})$. The second term is $O_p(N^\nu)$. The third term is $O_p(N^{1/2}T^{-1/2})$. Given the above result, we have

$$\|II_1\| = O_p(N^{1/2-\nu}T^{-3/2}) + O_p(T^{-1}) + O_p(N^{-\nu/2}T^{-1/2}).$$

Consider II_2.

$$\frac{1}{\sqrt{T}}\|II\|_2 = \frac{1}{N^\nu T^{3/2}}\Big\|\sum_{i=1}^N \widehat{\gamma}_i^2 \dot{e}_i \lambda_i' \dot{F}' \widehat{F} \hat{V}^{-1}\Big\| = \frac{1}{N^\nu T^{3/2}}\Big\|\sum_{i=1}^N \widehat{\gamma}_i^2 e_i \lambda_i' \dot{F}' \hat{F} \hat{V}^{-1}\Big\| \leq \frac{\|\sum_{i=1}^N \widehat{\gamma}_i^2 e_i \lambda_i'\|}{N^\nu \sqrt{T}} \frac{\|\dot{F}'\widehat{F}\|}{T}\|\widehat{V}^{-1}\|.$$

It suffices to investigate $\sum_{i=1}^N \widehat{\gamma}_i^2 e_i \lambda_i'$, which is equal to

$$\sum_{i\in\mathcal{I}_\phi} \gamma_i^2 e_i \lambda_i' + 2\sum_{i\in\mathcal{I}_\phi} \gamma_i \mathbf{u}_i e_i \lambda_i' + \sum_{i=1}^N \mathbf{u}_i^2 e_i \lambda_i'.$$

The first term is $\asymp_p N^{\nu/2}T^{1/2}$, and the second is $O_p(N^{\nu/2}) + O_p(N^\nu T^{-1/2})$, where the $O_p(N^\nu T^{-1/2})$ term is $\frac{1}{T}(\sum_{t=1}^T \|q_t\|^2)^{1/2}\sum_{i\in\mathcal{I}_\phi}\|\gamma_i\sigma_i^2\lambda_i\| \asymp_p N^\nu T^{-1/2}$. The third term is $O_p(N^{1/2}T^{-1})$. Given this, we have

$$\|II_2\| = O_p(N^{-\nu/2}) + O_p(T^{-1}) + O_p(N^{1/2-\nu}T^{-3/2}).$$

Consider II_3. Note that

$$\begin{aligned}\frac{1}{\sqrt{T}}\|II_3\| &= \frac{1}{N^\nu T^{3/2}}\Big\|\sum_{i=1}^N \widehat{\gamma}_i^2 \dot{e}_i e_i' \hat{F}\hat{V}^{-1}\Big\| \\ &= \frac{1}{N^\nu T^{3/2}}\Big\|\sum_{i=1}^N \widehat{\gamma}_i^2 \dot{e}_i e_i' (\hat{F} - \dot{F}R')\hat{V}^{-1}\Big\| + \frac{1}{N^\nu T^{3/2}}\Big\|\sum_{i=1}^N \widehat{\gamma}_i^2 \dot{e}_i e_i' \dot{F}R'\hat{V}^{-1}\Big\|.\end{aligned}$$

We use I_a and I_b to denote the above two expression. Further consider I_a. Ignore smaller order term, we see that

$$I_a \leq \frac{\|\sum_{i=1}^N \widehat{\gamma}_i^2 e_i e_i'\|_2}{N^\nu T}\frac{\|\widehat{F} - \dot{F}R'\|}{\sqrt{T}}\|\widehat{V}^{-1}\|.$$

However,

$$\|\sum_{i=1}^N \widehat{\gamma}_i^2 e_i e_i'\|_2 \leq \Big\|\sum_{i\in\mathcal{I}_\phi}\gamma_i^2 e_i e_i'\Big\|_2 + 2\Big\|\sum_{i\in\mathcal{I}_\phi}\gamma_i \mathbf{u}_i e_i e_i'\Big\|_2 + \Big\|\sum_{i=1}^N \mathbf{u}_i^2 e_i e_i'\Big\|_2$$

One can readily verify that the first term is $O_p(N^{\nu/2}T) + O_p(N^\nu)$, the second term is $O_p(N^{\nu/2}T^{1/2}) + O_p(N^\nu)$, and the third term is bounded by $2\|\sum_{i=1}^N \mathbf{u}_{i,1}^2 e_i e_i'\|_2 + 2\|\sum_{i=1}^N \mathbf{u}_{i,2}^2 e_i e_i'\|_2$. The former expression is bounded by $(\max_{i\leq N}\mathbf{u}_{i,1}^2)\|\sum_{i=1}^N e_i e_i'\|$, which, by the maximal inequality and Theorem 5.8 of Baik and Silverstein (2006), is $O_p(N^{1/2})$. The latter term is $O_p(N^{\nu/2}T^{-1/2}) +$

$O_p(N^\nu T^{-1})$. Summarizing all the results,

$$\Big\| \sum_{i=1}^{N} \widehat{\gamma}_i^2 e_i e_i' \Big\|_2 = O_p(N^{\nu/2}T) + O_p(N^\nu) + O_p(N^{1/2}).$$

So if $\frac{N^{1-\nu}}{T^2} \to 0$, we have

$$\frac{\| \sum_{i=1}^{N} \widehat{\gamma}_i^2 e_i e_i' \|_2}{N^\nu T} = o_p(1),$$

implying that term $\|I_a\|$ is negligible since it is of smaller order relative to $\frac{1}{\sqrt{T}}\|\widehat{F} - \dot{F}R'\|$. Further consider I_b. Similar arguments can show that

$$\|I_b\| = O_p(N^{-\nu/2}T^{-1/2}) + O_p(T^{-1}) + O_p(\frac{1}{N^\nu}\sqrt{\frac{N}{T^2}}) + O_p(\frac{1}{N^\nu}\frac{N}{T^2}).$$

where term $O_p(\frac{1}{N^\nu}\frac{N}{T^2})$ is equal to

$$\frac{1}{N^\nu T^{7/2}}\Big(\sum_{t=1}^{T}\|q_t\|^2\Big)\Big(\sum_{i=1}^{N}\sigma_i^4\Big)\Big(\sum_{t=1}^{T}\|f_t\|^2\Big)^{1/2} \asymp_p \frac{N^{1-\nu}}{T^2}.$$

Summarizing the results of II_1, II_2 and II_3, we conclude that

$$\frac{\|\widehat{F} - \dot{F}R'\|}{\sqrt{T}} = O_p(N^{-\nu/2}) + O_p(T^{-1}) + O_p(\frac{1}{N^\nu}\sqrt{\frac{N}{T^2}}) + O_p(\frac{1}{N^\nu}\frac{N}{T^2}).$$

Under the assumption that $\frac{N^{1-\nu}}{T^2} \to 0$, by the fact

$$\frac{1}{N^\nu}\sqrt{\frac{N}{T^2}} = N^{-\nu/2}\sqrt{\frac{N^{1-\nu}}{T^2}} = o(N^{-\nu/2}),$$

we therefore have

$$\frac{1}{\sqrt{T}}\|\widehat{F} - \dot{F}R'\| = O_p(N^{-\nu/2}) + O_p(T^{-1}) + O_p(\frac{N^{1-\nu}}{T^2}).$$

The above proof also confirms that all the three terms on right hand side are not $o_p(\cdot)$ terms. This completes the proof. $\square$

Lemma 3. *For the PCA method, under Assumptions 1-6, together with* $\frac{N^{1-\nu}}{T} \geq c$*, the estimated factor*

$\widehat{F}^{\text{PCA}}$ *is not a consistent estimator F in the sense that for any invertible matrix R_{PCA},*

$$\frac{1}{\sqrt{T}}\|\widehat{F}^{\text{PCA}} - \dot{F}R'_{\text{PCA}}\| \geq c^*$$

for some constant c^* with a strictly positive probability.

Proof of Lemma 3. Similarly as the analysis in the previous preposition, we first investigate the magnitude of the eigenvalues for the data $\sum_{i=1}^{N} \dot{X}_i \dot{X}_i'$ in the PCA. Let $\widetilde{F}$ and $\widetilde{V}$ be the eigenvectors and eigenvalues of the matrix $\sum_{i=1}^{N} \dot{X}_i \dot{X}_i'$ (note that $\widetilde{F}$ and $\widetilde{V}$ are different from the the same symbols in the previous lemma), we therefore have

$$\widetilde{V} = \widetilde{F}'\left[\dot{F}\sum_{i=1}^{N}\lambda_i\lambda_i'\dot{F}' + \dot{F}\sum_{i=1}^{N}\lambda_i\dot{e}_i' + \sum_{i=1}^{N}\dot{e}_i\lambda_i'\dot{F}' + \sum_{i=1}^{N}\dot{e}_i\dot{e}_i'\right]\widetilde{F} = I_1 + \cdots + I_4, \quad \text{say.}$$

Consider I_1. Note that $\widetilde{F}'\dot{F} = O_p(\sqrt{T})$ and $\sum_{i=1}^{N}\lambda_i\lambda_i' = O(N^\nu)$. Given this, we have $I_1 = O_p(N^\nu T)$. Next consider I_2, which is bounded in norm by

$$\|\widetilde{F}'\dot{F}\|\Big[\sum_{t=1}^{T}\Big\|\sum_{i=1}^{N}\lambda_i e_{it}\Big\|^2\Big]^{1/2}\Big[\sum_{t=1}^{T}\|\widetilde{f}_t\|^2\Big]^{1/2} = O_p(N^{\nu/2}T).$$

The third term is the transpose of the second and is therefore $O_p(N^{\nu/2}T)$. Consider I_4, which is equal to

$$I_4 = \widetilde{F}'\sum_{i=1}^{N}e_ie_i'\widetilde{F} = \sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}[e_{it}e_{is} - E(e_{it}e_{is})] + \sum_{t=1}^{T}\sum_{s=1}^{T}\widetilde{f}_t\widetilde{f}_s'\sum_{i=1}^{N}E(e_{it}e_{is}).$$

The first expression on right hand side is bounded in norm by

$$\Big[\sum_{i=1}^{N}\|\widetilde{f}_t\|^2\Big]\Big[\sum_{t=1}^{T}\sum_{s=1}^{T}\Big|\sum_{i=1}^{N}[e_{it}e_{is} - E(e_{it}e_{is})]\Big|^2\Big]^{1/2} = O_p(\sqrt{NT}).$$

The second expression is $O_p(N)$. Given this, we have

$$\widetilde{V} = O_p(N^\nu T) + O_p(\sqrt{NT}) + O_p(N).$$

Note that, when $\frac{N^{1-\nu}}{T} > c$, the last term is either of the same magnitude with the first term, or dominates the first term. This means that

$$\frac{1}{N^{\nu}T}\left|\widetilde{V}-\widetilde{F}'\dot{F}\sum_{i=1}^{N}\lambda_i\lambda_i'\dot{F}\widetilde{F}\right|>c^{\bullet}$$

for some $c^{\bullet}$ with a positive probability, implying that even with some approximate normalization, $\widetilde{V}$ is no longer a good estimator for the largest two eigenvalues of the matrix $\dot{F}\sum_{i=1}^{N}\lambda_i\lambda_i'\dot{F}'$. Since $\frac{1}{\sqrt{T}}F$ is the eigenvectors of the matrix $\dot{F}\sum_{i=1}^{N}\lambda_i\lambda_i'\dot{F}'$ because $\frac{1}{T}\dot{F}'\dot{F}=I$ and $\sum_{i=1}^{N}\lambda_i\lambda_i'$ is a diagonal matrix, we immediately obtain that the estimator factor $\widehat{F}=\sqrt{T}\widetilde{F}$ is no longer a consistent estimator of F in the sense that $\frac{1}{\sqrt{T}}\|\widehat{F}-\dot{F}R'\|\geq c^*$ for some constant c^* with a strictly positive probability. We can use contradiction arguments to show this. Suppose that this is not the case, we therefore have $\frac{1}{\sqrt{T}}\|\widehat{F}-\dot{F}R'\|=o_p(1)$. Because $\frac{1}{T}\dot{F}'\dot{F}=I$ and $\frac{1}{T}\widehat{F}'\widehat{F}=I$, we see that R is an orthonormal matrix. This result would imply that $\frac{1}{N^{\nu}T}\left|\widetilde{V}-\widetilde{F}'\dot{F}\sum_{i=1}^{N}\lambda_i\lambda_i'\dot{F}\widetilde{F}\right|=o_p(1)$. A contradiction is obtained. This completes the proof. $\square$

Lemma 4. *Under Assumptions 1-6, if $\frac{N^{1-\nu}}{T}\to 0$, for the PCA method, we have*

$$\frac{1}{\sqrt{T}}\|\widehat{F}^{\mathrm{PCA}}-\dot{F}R_{\mathrm{PCA}}'\|\asymp_p N^{-\nu/2}+\frac{N^{1-\nu}}{T}.$$

Proof of Lemma 4. We use $\widehat{F}$ and R to denote the estimated factors and the rotational matrix in the PCA method for notational simplicity, if no confusion arise. By definition, we have

$$\left[\frac{1}{N^{\nu}T}\dot{F}\sum_{i=1}^{N}\lambda_i\lambda_i'\dot{F}'+\frac{1}{N^{\nu}T}\dot{F}\sum_{i=1}^{N}\lambda_i\dot{e}_i+\frac{1}{N^{\nu}T}\sum_{i=1}^{N}\dot{e}_i\lambda_i'\dot{F}'+\frac{1}{N^{\nu}T}\sum_{i=1}^{N}\dot{e}_i\dot{e}_i'\right]\widehat{F}=\widehat{F}\widehat{V}.$$

Given $\frac{N^{1-\nu}}{T}\to 0$, by the proof of Lemma 3, we see $\widehat{V}-N^{-\nu}\Lambda'\Lambda=o_p(1)$, implying $\widehat{V}^{-1}=O_p(1)$. Thus,

$$\widehat{F}-\dot{F}R'=\frac{1}{N^{\nu}T}\dot{F}\sum_{i=1}^{N}\lambda_i\dot{e}_i\widehat{F}\widehat{V}^{-1}+\frac{1}{N^{\nu}T}\sum_{i=1}^{N}\dot{e}_i\lambda_i'\dot{F}'\widehat{F}\widehat{V}^{-1}+\frac{1}{N^{\nu}T}\sum_{i=1}^{N}\dot{e}_i\dot{e}_i'\widehat{F}\widehat{V}^{-1}\equiv I_1+I_2+I_3.$$

where $R=\frac{1}{N^{\nu}T}\widehat{V}^{-1}\widehat{F}'\dot{F}\Lambda'\Lambda$. We consider the three terms on right hand side one by one. Consider

I_1.

$$\frac{1}{\sqrt{T}}\|I_1\| \leq \frac{1}{\sqrt{T}}\Big\|\frac{1}{N^\nu T}\dot{F}\sum_{i=1}^{N}\lambda_i e_i(\widehat{F}-\dot{F}R')\widehat{V}^{-1}\Big\| + \frac{1}{\sqrt{T}}\Big\|\frac{1}{N^\nu T}\dot{F}\sum_{i=1}^{N}\lambda_i e_i\dot{F}\widehat{V}^{-1}\Big\|$$
$$\leq \frac{1}{\sqrt{N^\nu}}\frac{\|\dot{F}\|}{\sqrt{T}}\frac{\|\sum_{i=1}^{N}\lambda_i e_i\|}{\sqrt{N^\nu T}}\frac{\|\widehat{F}-\dot{F}R'\|}{\sqrt{T}}\|\widehat{V}^{-1}\| + \frac{1}{\sqrt{N^\nu T}}\frac{\|\dot{F}\|}{\sqrt{T}}\frac{\|\sum_{i=1}^{N}\lambda_i e_i\dot{F}\|}{\sqrt{N^\nu T}}\|\widehat{V}^{-1}\|$$

The first term is a smaller order one relative to $\frac{1}{\sqrt{T}}\|\widehat{F}-\dot{F}R'\|$ and therefore is negligible. The second term is $O_p(N^{-\nu/2}T^{-1/2})$. So we have $\frac{1}{\sqrt{T}}\|I_1\| = O_p(N^{-\nu/2}T^{-1/2})$. Consider I_2.

$$\frac{1}{\sqrt{T}}\|I_2\| \leq \frac{1}{\sqrt{N^\nu}}\frac{\|\sum_{i=1}^{N}\lambda_i e_i\|}{\sqrt{N^\nu T}}\frac{\|\dot{F}\|}{\sqrt{T}}\frac{\|\widehat{F}\|}{\sqrt{T}}\|\widehat{V}^{-1}\| = O_p(N^{-\nu/2}).$$

Next consider the third term. Ignore the smaller order term, we see that

$$\frac{1}{\sqrt{T}}\|I_3\| \leq \frac{1}{\sqrt{T}}\Big\|\frac{1}{N^\nu T}\sum_{i=1}^{N}e_i e_i'(\widehat{F}-\dot{F}R')\widehat{V}^{-1}\Big\| + \frac{1}{\sqrt{T}}\Big\|\frac{1}{N^\nu T}\sum_{i=1}^{N}e_i e_i'\dot{F}R'\widehat{V}^{-1}\Big\|$$
$$\leq \Big\|\frac{1}{N^\nu T}\sum_{i=1}^{N}e_i e_i'\Big\|_2\frac{\|\widehat{F}-\dot{F}R'\|}{\sqrt{T}}\|\widehat{V}^{-1}\| + \frac{1}{\sqrt{T}}\Big\|\frac{1}{N^\nu T}\sum_{i=1}^{N}[e_i e_i' - E(e_i e_i')]\dot{F}R'\widehat{V}^{-1}\Big\|$$
$$+ \frac{1}{\sqrt{T}}\Big\|\frac{1}{N^\nu T}\sum_{i=1}^{N}E(e_i e_i')\dot{F}R'\widehat{V}^{-1}\Big\|$$

where $\|\cdot\|_2$ denotes the spectral norm. By Theorem 5.8 of Baik and Silverstein (2006), we have that $\|\frac{1}{N^\nu T}\sum_{i=1}^{N}e_i e_i'\|_2 = O_p(N^{-\nu})$, so the first term is a smaller order one relative to $\frac{1}{\sqrt{T}}\|\widehat{F}-\dot{F}R'\|$ and therefore is negligible. The second term can be readily verified to be $O_p(N^{-\nu/2}T^{-1/2})$ and the third term is $O_p(N^{1-\nu}T^{-1})$.

With the results on I_1, I_2 and I_3, we therefore have

$$\frac{1}{\sqrt{T}}\|\widehat{F}-\dot{F}R'\| = O_p(N^{-\nu/2}) + O_p(N^{1-\nu}T^{-1}).$$

It is easy to verify that the two terms on right hand side are not $o_p(\cdot)$ terms. Given this, we have Lemma 4. This completes the proof. □

Proof of Proposition 1. Proposition 1 is the direct result of Lemmas 1 – 4. This completes the proof. □.

Proof of Proposition 2. We prove the results with two cases. Case one: $\frac{N^{1-\nu}}{T} \geq c$ for some $c > 0$; and Case two: $\frac{N^{1-\nu}}{T} \to 0$.

Case one: According to Proposition 1, when $\frac{N^{1-\nu}}{T} \geq c$ for some $c > 0$, the estimates of the factors in the sPCA are consistent but the estimates in the PCA are not. As the result, the correlation coefficient of the estimated factors and the true factors in the sPCA converges to 1, but the corresponding correlation coefficient in the PCA is strictly less than 1. In the proof of Lemma 5 below, we show that when a single factor is used to predict, the asymptotic MSFE is equal to

$$\text{MSFE} = \beta^2(1 - r^2) + \sigma_\epsilon^2$$

where r is the correlation coefficient. Now it is easy to see that the higher the correlation, the better the forecast. So the sPCA forecast outperforms the PCA forecast. In the current setting, we have more factors used in forecast. However, with the same arguments in Lemma 5, we can show that

$$\text{MSFE} - \sigma_\epsilon^2 \asymp_p \frac{1}{T}\|\widehat{F} - \dot{F}R\|^2.$$

Given that $\frac{1}{T}\|\widehat{F} - \dot{F}R\|^2 = o_p(1)$ in the sPCA and $\frac{1}{T}\|\widehat{F} - \dot{F}R\|^2 \geq c$ with a positive probability in the PCA, we immediately obtain that the sPCA has a superior forecast performance than the PCA.

Case two: According to the result of Proposition 3, we see that the MSFEs of the two methods are

$$\text{MSFE}_{\text{PCA}} = \frac{1}{T}3\sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\beta^{\star\prime}(\Lambda'\Lambda)^{-1}\Gamma_t^{\text{PCA}}(\Lambda'\Lambda)^{-1}\beta^\star,$$
$$\text{MSFE}_{\text{sPCA}} = \frac{1}{T}3\sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\beta^{\star\prime}(\Lambda'W\Lambda)^{-1}\Lambda'\Gamma_t^{\text{sPCA}}\Lambda(\Lambda'W\Lambda)^{-1}\beta^\star.$$

However, one can readily verify that

$$\frac{1}{T}\sum_{t=1}^{T}\beta^{\star\prime}(\Lambda'\Lambda)^{-1}\Gamma_t^{\text{PCA}}(\Lambda'\Lambda)^{-1}\beta^\star \asymp_p \left(\frac{N^{1-\nu}}{T}\right)^2$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\beta^{\star\prime}(\Lambda' W\Lambda)^{-1}\Lambda'\Gamma_t^{\text{sPCA}}\Lambda(\Lambda' W\Lambda)^{-1}\beta^{\star} \asymp_p N^{-\nu}.$$

So under the assumption $\frac{N^{1-\nu/2}}{T} \to \infty$, we see that the MSFE of the PCA is larger than that of the sPCA. This completes the whole proof. □

B Theoretical proofs for the case of strong factors

Proof of Proposition 3. The proof of Proposition 3 is based on Theorem 3 of Bai and Ng (2006). We only highlight some differences. The whole arguments are essentially the same. Here we use the PCA method to illustrate. With some straightforward computations (see also Bai (2003), we would have that under $\sqrt{N}/T \to 0$,

$$\sqrt{N}(\widehat{f}_t - R_{\text{PCA}}\dot{f}_t) = \widehat{V}^{-1}\frac{1}{T}\widehat{F}'F\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{it} + o_p(1).$$

where $R_{\text{PCA}} = \frac{1}{NT}\widehat{V}^{-1}\widehat{F}'\dot{F}\Lambda'\Lambda$. In Bai (2003), he first shows the probability limit of $\widehat{V}$ and $\frac{1}{T}\widehat{F}'\dot{F}$. With these results, he next derives the final limiting distribution.

The treatment of this paper is slightly different. We rewrite the above display as

$$\sqrt{N}(\widehat{f}_t - R_{\text{PCA}}\dot{f}_t) = R_{\text{PCA}}\left(\frac{1}{N}\Lambda'\Lambda\right)^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{it} + o_p(1).$$

The benefits of the above display is that it involves the rotational matrix and the true values and we can see clearly the consequence of rotational indeterminacy. With the arguments in Bai and Ng (2006), we have this proposition. □

To prove Proposition 4, we first present two lemmas on the MSFEs of the sPCA and PCA forecasts, of which the associated proofs can be found in Huang, Jiang, Li, Tong, and Zhou (2021).

Lemma 5. *Let ξ° and θ° be*

$$\xi^\circ = \frac{\sum_{i=1}^N \phi_i^\circ \psi_i^\circ}{\sum_{i=1}^N (\phi_i^{\circ 2} - \psi_i^{\circ 2})} = \frac{\sum_{i=1}^N \phi_i^3 \psi_i}{\sum_{i=1}^N \phi_i^2 (\phi_i^2 - \psi_i^2)}, \tag{A8}$$

and

$$\theta^\circ = \begin{cases} \frac{1+\sqrt{1+4\xi^{\circ 2}}}{\sqrt{4\xi^{\circ 2}+(1+\sqrt{1+4\xi^{\circ 2}})^2}} & \text{if } \sum_{i=1}^N (\phi_i^{\circ 2} - \psi_i^{\circ 2}) > 0; \\ \frac{\sqrt{1+4\xi^{\circ 2}}-1}{\sqrt{4\xi^{\circ 2}+(1-\sqrt{1+4\xi^{\circ 2}})^2}} & \text{if } \sum_{i=1}^N (\phi_i^{\circ 2} - \psi_i^{\circ 2}) < 0. \end{cases} \tag{A9}$$

If one uses the first principal component of $\{\widehat{\gamma}_i X_{i,t}\}$ to conduct forecasting, under Assumptions 1–5, as $N \to \infty, T \to \infty$ and $\sqrt{N}/T \to 0$, then the asymptotic MSFE is

$$\text{MSFE}_{\text{sPCA}} = \beta^2(1-\theta^{\circ 2}) + \sigma_\epsilon^2, \ \textit{where} \ \sigma_\epsilon^2 = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \epsilon_{t+h}^2. \tag{A10}$$

Lemma 6. *Let $\xi = \sum_{i=1}^N \phi_i \psi_i \Big/ \sum_{i=1}^N (\phi_i^2 - \psi_i^2)$ and*

$$\theta = \begin{cases} \frac{1+\sqrt{1+4\xi^2}}{\sqrt{4\xi^2+(1+\sqrt{1+4\xi^2})^2}} & \text{if } \sum_{i=1}^N (\phi_i^2 - \psi_i^2) > 0; \\ \frac{\sqrt{1+4\xi^2}-1}{\sqrt{4\xi^2+(1-\sqrt{1+4\xi^2})^2}} & \text{if } \sum_{i=1}^N (\phi_i^2 - \psi_i^2) < 0. \end{cases} \tag{A11}$$

If one uses the first principal component of $\{X_{i,t}\}$ to conduct forecasting, under Assumptions 1–5, as $N \to \infty$ and $T \to \infty$, then the asymptotic MSFE is

$$\text{MSFE}_{\text{PCA}} = \beta^2(1-\theta^2) + \sigma_\epsilon^2 \ \textit{where} \ \sigma_\epsilon^2 = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \epsilon_{t+h}^2. \tag{A12}$$

Proof of Proposition 4. First consider the case of $|\phi_i| > |\psi_i|$ and $\phi_i \psi_i \geq 0$ for all i. It is no loss of generality to assume $\phi_i > \psi_i \geq 0$. Otherwise, do the manipulation of $\phi_i \to -\phi_i$ and $\psi_i \to -\psi_i$. Consider the case that $N = 2$, we need to show

$$\frac{\phi_1^2 \phi_1 \psi_1 + \phi_2^2 \phi_2 \psi_2}{\phi_1^2(\phi_1^2 - \psi_1^2) + \phi_2^2(\phi_2^2 - \psi_2^2)} \leq \frac{\phi_1 \psi_1 + \phi_2 \psi_2}{(\phi_1^2 - \psi_1^2) + (\phi_2^2 - \psi_2^2)}. \tag{A13}$$

Straightforward computations indicate that the above inequality is equivalent to

$$(\phi_1^2-\phi_2^2)\Big(\frac{\phi_1\psi_1}{\phi_1^2-\psi_1^2}-\frac{\phi_2\psi_2}{\phi_2^2-\psi_2^2}\Big)\le 0. \tag{A14}$$

It suffices to verify that $\frac{\phi\psi}{\phi^2-\psi^2}$ deceases as ϕ^2 increases. Since we normalize $\dot{X}_{it}$, we have $\phi_i^2+\psi_i^2+\sigma_e^2=1$ for all i. So a larger ϕ^2 leads to a smaller ψ^2, which further leads to a smaller $\frac{\phi\psi}{\phi^2-\psi^2}=\frac{z}{1-z^2}$ with $z=\psi/\phi$. Given this fact, we have shown the result for $N=2$. Now we are to prove the general case by induction. Suppose that the result holds for $N=N^*-1$, i.e.,

$$\frac{\sum_{i=1}^{N^*-1}\phi_i^2\phi_i\psi_i}{\sum_{i=1}^{N^*-1}\phi_i^2(\phi_i^2-\psi_i^2)}\le\frac{\sum_{i=1}^{N^*-1}\phi_i\psi_i}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)}. \tag{A15}$$

Consider the case $N=N^*$. Among ϕ_i for $i=1,2,\ldots,N^*$, there must exist a largest ϕ (which may be not unique). It is no loss of generality to assume that ϕ_{N^*} is the largest. Otherwise, we change the positions of the largest ϕ_i and ϕ_{N^*}. From (A15), we have

$$\underbrace{\frac{\sum_{i=1}^{N^*-1}\phi_i^2\phi_i\psi_i}{\sum_{i=1}^{N^*-1}\phi_i\psi_i}}_{\equiv\mathbf{a}}\le\underbrace{\frac{\sum_{i=1}^{N^*-1}\phi_i^2(\phi_i^2-\psi_i^2)}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)}}_{\equiv\mathbf{b}}\le\phi_{N^*}^2, \tag{A16}$$

where $\mathbf{a}$ and $\mathbf{b}$ are implicitly defined above, and the second inequality is due to the fact that $\phi_{N^*}^2$ is the largest. Note that we need prove

$$\frac{\sum_{i=1}^{N^*}\phi_i^2\phi_i\psi_i}{\sum_{i=1}^{N^*}\phi_i^2(\phi_i^2-\psi_i^2)}\le\frac{\sum_{i=1}^{N^*}\phi_i\psi_i}{\sum_{i=1}^{N^*}(\phi_i^2-\psi_i^2)}, \tag{A17}$$

which is equivalent to

$$\frac{\sum_{i=1}^{N^*-1}\phi_i^2\phi_i\psi_i+\phi_{N^*}^2\phi_{N^*}\psi_{N^*}}{\sum_{i=1}^{N^*-1}\phi_i^2(\phi_i^2-\psi_i^2)+\phi_{N^*}^2(\phi_{N^*}^2-\psi_{N^*}^2)}\le\frac{\sum_{i=1}^{N^*-1}\phi_i\psi_i+\phi_{N^*}\psi_{N^*}}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)+(\phi_{N^*}^2-\psi_{N^*}^2)}. \tag{A18}$$

According to the definitions of $\mathbf{a}$ and $\mathbf{b}$, the above inequality is identical to

$$\frac{\mathbf{a}\sum_{i=1}^{N^*-1}\phi_i\psi_i+\phi_{N^*}^2\phi_{N^*}\psi_{N^*}}{\mathbf{b}\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)+\phi_{N^*}^2(\phi_{N^*}^2-\psi_{N^*}^2)}\le\frac{\sum_{i=1}^{N^*-1}\phi_i\psi_i+\phi_{N^*}\psi_{N^*}}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)+(\phi_{N^*}^2-\psi_{N^*}^2)}. \tag{A19}$$

Because $\mathbf{a} \leq \mathbf{b}$, it is seen that

$$\frac{\mathbf{a}\sum_{i=1}^{N^*-1}\phi_i\psi_i + \phi_{N^*}^2\phi_{N^*}\psi_{N^*}}{\mathbf{b}\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2) + \phi_{N^*}^2(\phi_{N^*}^2-\psi_{N^*}^2)} \leq \frac{\mathbf{b}\sum_{i=1}^{N^*-1}\phi_i\psi_i + \phi_{N^*}^2\phi_{N^*}\psi_{N^*}}{\mathbf{b}\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2) + \phi_{N^*}^2(\phi_{N^*}^2-\psi_{N^*}^2)}. \tag{A20}$$

So we remain to prove

$$\frac{\mathbf{b}\sum_{i=1}^{N^*-1}\phi_i\psi_i + \phi_{N^*}^2\phi_{N^*}\psi_{N^*}}{\mathbf{b}\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2) + \phi_{N^*}^2(\phi_{N^*}^2-\psi_{N^*}^2)} \leq \frac{\sum_{i=1}^{N^*-1}\phi_i\psi_i + \phi_{N^*}\psi_{N^*}}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2) + (\phi_{N^*}^2-\psi_{N^*}^2)}. \tag{A21}$$

Some straightforward computations show that the above inequality is equivalent to

$$(\phi_{N^*}^2 - \mathbf{b})\Big(\frac{\phi_{N^*}\psi_{N^*}}{\phi_{N^*}^2-\psi_{N^*}^2} - \frac{\sum_{i=1}^{N^*-1}\phi_i\psi_i}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)}\Big) \leq 0. \tag{A22}$$

Let $z_i = \psi_i/\phi_i$. Note that ϕ_{N^*} is the largest, which means for each i,

$$\frac{\phi_{N^*}\psi_{N^*}}{\phi_{N^*}^2-\psi_{N^*}^2} = \frac{z_{N^*}}{1-z_{N^*}^2} \leq \frac{z_i}{1-z_i^2} = \frac{\phi_i\psi_i}{\phi_i^2-\psi_i^2}. \tag{A23}$$

With the fact that $\frac{a}{b} \leq \frac{\sum_i a_i}{\sum_i b_i}$ if $\frac{a}{b} \leq \frac{a_i}{b_i}$ for each i, we have

$$\frac{\phi_{N^*}\psi_{N^*}}{\phi_{N^*}^2-\psi_{N^*}^2} - \frac{\sum_{i=1}^{N^*-1}\phi_i\psi_i}{\sum_{i=1}^{N^*-1}(\phi_i^2-\psi_i^2)} \leq 0. \tag{A24}$$

So we obtain (A22) because $\phi_{N^*}^2 \geq \mathbf{b}$. This proves the result of the first case. The remaining three cases can be proved by the same argument and the details are therefore omitted.

Now consider the first two cases, which suggest that $\sum_{i=1}^N \phi_i^2 > \sum_{i=1}^N \psi_i^2$ and $|\xi^\circ| \leq |\xi|$. According to the formulas of MSFE and θ, we see that the MSFE is a decreasing function of θ^2 and θ^2 is a deceasing function of $|\xi|$. In order to make MSFE smaller, we should require $|\xi|$ to be smaller. So the sPCA outperforms the PCA in the former two cases. Next consider the later two cases, in which we have $\sum_{i=1}^N \phi_i^2 < \sum_{i=1}^N \psi_i^2$ and $|\xi^\circ| \geq |\xi|$. By checking the formulas, we find that the MSFE is a decreasing function of θ^2 but θ^2 is a increasing function of $|\xi|$. Given this fact, we conclude that the sPCA also outperforms in the later two cases. This completes the proof of Proposition 4. □

C Data Appendix

This appendix first lists the 123 macroeconomic time series considered and obtained from the Federal Reserve Monthly Database for Economic Research (FRED-MD). For each variable, we report the FRED-MD mnemonics, a full variable description, and the transformation code (trcode) used to ensure stationarity of the underlying data series. The particular form of the transformations are specified below. To fix notation, let $x_{i,t}^{\text{raw}}$ and $x_{i,t}^{\text{tr}}$ denote the raw and transformed version of the ith variable observed at time t, respectively, and let $\Delta = (1 - L)$, with a lag operator $Lx_{i,t}^{\text{raw}} = x_{i,t-1}^{\text{raw}}$. We then apply one of seven possible transformations:

1. lvl: $x_{i,t}^{\text{tr}} = x_{i,t}^{\text{raw}}$
2. Δ lvl: $x_{i,t}^{\text{tr}} = x_{i,t}^{\text{raw}} - x_{i,t-1}^{\text{raw}}$
3. Δ^2 lvl: $x_{i,t}^{\text{tr}} = \Delta^2 x_{i,t}^{\text{raw}}$
4. ln: $x_{i,t}^{\text{tr}} = \ln\left(x_{i,t}^{\text{raw}}\right)$
5. Δ ln: $x_{i,t}^{\text{tr}} = \ln\left(x_{i,t}^{\text{raw}}\right) - \ln\left(x_{i,t-1}^{\text{raw}}\right)$
6. Δ^2 ln: $x_{i,t}^{\text{tr}} = \Delta^2 \ln\left(x_{i,t}^{\text{raw}}\right)$
7. $\Delta\ \frac{lvl_t - lvl_{t-1}}{lvl_{t-1}}$: $x_{i,t}^{\text{tr}} = \Delta\left(x_{i,t}^{\text{raw}} / x_{i,t-1}^{\text{raw}} - 1\right)$

No.	Mnemonic	Variable description	trcode
1	RPI	Real Personal Income	5
2	W875RX1	Real personal income ex transfer receipts	5
3	INDPRO	IP Index	5
4	IPFPNSS	IP: Final Products and Nonindustrial Supplies	5
5	IPFINAL	IP: Final Products (Market Group)	5
6	IPCONGD	IP: Consumer Goods	5
7	IPDCONGD	IP: Durable Consumer Goods	5
8	IPNCONGD	IP: Nondurable Consumer Goods	5
9	IPBUSEQ	IP: Business Equipment	5
10	IPMAT	IP: Materials	5
11	IPDMAT	IP: Durable Materials	5
12	IPNMAT	IP: Nondurable Materials	5
13	IPMANSICS	IP: Manufacturing (SIC)	5
14	IPB51222s	IP: Residential Utilities	5
15	IPFUELS	IP: Fuels	5
16	CUMFNS	Capacity Utilization: Manufacturing	2
17	HWI	Help-Wanted Index for United States	2
18	HWIURATIO	Ratio of Help Wanted/No. Unemployed	2
19	CLF16OV	Civilian Labor Force	5
20	CE16OV	Civilian Employment	5
21	UNRATE	Civilian Unemployment Rate	2
22	UEMPMEAN	Average Duration of Unemployment (Weeks)	2
23	UEMPLT5	Civilians Unemployed - Less Than 5 Weeks	5
24	UEMP5TO14	Civilians Unemployed for 5-14 Weeks	5
25	UEMP15OV	Civilians Unemployed - 15 Weeks & Over	5
26	UEMP15T26	Civilians Unemployed for 15-26 Weeks	5
27	UEMP27OV	Civilians Unemployed for 27 Weeks and Over	5
28	CLAIMSx	Initial Claims	5
29	PAYEMS	All Employees: Total nonfarm	5
30	USGOOD	All Employees: Goods-Producing Industries	5
31	CES1021000001	All Employees: Mining and Logging: Mining	5
32	USCONS	All Employees: Construction	5
33	MANEMP	All Employees: Manufacturing	5
34	DMANEMP	All Employees: Durable goods	5
35	NDMANEMP	All Employees: Nondurable goods	5
36	SRVPRD	All Employees: Service-Providing Industries	5
37	USTPU	All Employees: Trade, Transportation & Utilities	5
38	USWTRADE	All Employees: Wholesale Trade	5
39	USTRADE	All Employees: Retail Trade	5
40	USFIRE	All Employees: Financial Activities	5
41	USGOVT	All Employees: Government	5
42	CES0600000007	Avg Weekly Hours: Goods-Producing	1
43	AWOTMAN	Avg Weekly Overtime Hours: Manufacturing	2
44	AWHMAN	Avg Weekly Hours: Manufacturing	1
45	CES0600000008	Avg Hourly Earnings: Goods-Producing	6
46	CES2000000008	Avg Hourly Earnings: Construction	6
47	CES3000000008	Avg Hourly Earnings: Manufacturing	6
48	HOUST	Housing Starts: Total New Privately Owned	4
49	HOUSTNE	Housing Starts, Northeast	4
50	HOUSTMW	Housing Starts, Midwest	4
51	HOUSTS	Housing Starts, South	4
52	HOUSTW	Housing Starts, West	4
53	PERMIT	New Private Housing Permits (SAAR)	4
54	PERMITNE	New Private Housing Permits, Northeast (SAAR)	4
55	PERMITMW	New Private Housing Permits, Midwest (SAAR)	4
56	PERMITS	New Private Housing Permits, South (SAAR)	4
57	PERMITW	New Private Housing Permits, West (SAAR)	4
58	DPCERA3M086SBEA	Real personal consumption expenditures	5
59	CMRMTSPLx	Real Manu. and Trade Industries Sales	5
60	RETAILx	Retail and Food Services Sales	5
61	AMDMNOx	New Orders for Durable Goods	5

No.	Mnemonic	Industry Description	Category
62	AMDMUOx	Unfilled Orders for Durable Goods	5
63	BUSINVx	Total Business Inventories	5
64	ISRATIOx	Total Business: Inventories to Sales Ratio	2
65	M1SL	M1 Money Stock	6
66	M2SL	M2 Money Stock	6
67	M2REAL	Real M2 Money Stock	5
68	AMBSL	St. Louis Adjusted Monetary Base	6
69	TOTRESNS	Total Reserves of Depository Institutions	6
70	NONBORRES	Reserves Of Depository Institutions	7
71	BUSLOANS	Commercial and Industrial Loans	6
72	REALLN	Real Estate Loans at All Commercial Banks	6
73	NONREVSL	Total Nonrevolving Credit	6
74	CONSPI	Nonrevolving consumer credit to Personal Income	2
75	MZMSL	MZM Money Stock	6
76	DTCOLNVHFNM	Consumer Motor Vehicle Loans Outstanding	6
77	DTCTHFNM	Total Consumer Loans and Leases Outstanding	6
78	INVEST	Securities in Bank Credit at All Commercial Banks	6
79	FEDFUNDS	Effective Federal Funds Rate	2
80	CP3Mx	3-Month AA Financial Commercial Paper Rate	2
81	TB3MS	3-Month Treasury Bill	2
82	TB6MS	6-Month Treasury Bill	2
83	GS1	1-Year Treasury Rate	2
84	GS5	5-Year Treasury Rate	2
85	GS10	10-Year Treasury Rate	2
86	AAA	Moody's Seasoned Aaa Corporate Bond Yield	2
87	BAA	Moody's Seasoned Baa Corporate Bond Yield	2
88	COMPAPFFx	3-Month Commercial Paper Minus FEDFUNDS	1
89	TB3SMFFM	3-Month Treasury C Minus FEDFUNDS	1
90	TB6SMFFM	6-Month Treasury C Minus FEDFUNDS	1
91	T1YFFM	1-Year Treasury C Minus FEDFUNDS	1
92	T5YFFM	5-Year Treasury C Minus FEDFUNDS	1
93	T10YFFM	10-Year Treasury C Minus FEDFUNDS	1
94	AAAFFM	Moody's Aaa Corporate Bond Minus FEDFUNDS	1
95	BAAFFM	Moody's Baa Corporate Bond Minus FEDFUNDS	1
96	EXSZUSx	Switzerland/U.S. Foreign Exchange Rate	5
97	EXJPUSx	Japan/U.S. Foreign Exchange Rate	5
98	EXUSUKx	U.S./U.K. Foreign Exchange Rate	5
99	EXCAUSx	Canada/U.S. Foreign Exchange Rate	5
100	PPIFGS	PPI: Finished Goods	6
101	PPIFCG	PPI: Finished Consumer Goods	6
102	PPIITM	PPI: Intermediate Materials	6
103	PPICRM	PPI: Crude Materials	6
104	OILPRICEx	Crude Oil, spliced WTI and Cushing	6
105	PPICMM	PPI: Metals and metal products:	6
106	CPIAUCSL	CPI: All Items	6
107	CPIAPPSL	CPI: Apparel	6
108	CPITRNSL	CPI: Transportation	6
109	CPIMEDSL	CPI: Medical Care	6
110	CUSR0000SAC	CPI: Commodities	6
111	CUUR0000SAD	CPI: Durables	6
112	CUSR0000SAS	CPI: Services	6
113	CPIULFSL	CPI: All Items Less Food	6
114	CUUR0000SA0L2	CPI: All items less shelter	6
115	CUSR0000SA0L5	CPI: All items less medical care	6
116	PCEPI	Personal Cons. Expend.: Chain Index	6
117	DDURRG3M086SBEA	Personal Cons. Exp: Durable goods	6
118	DNDGRG3M086SBEA	Personal Cons. Exp: Nondurable goods	6
119	DSERRG3M086SBEA	Personal Cons. Exp: Services	6
120	S&P 500	S&P's Common Stock Price Index: Composite	5
121	S&P: indust	S&P's Common Stock Price Index: Industrials	5
122	S&P div yield	S&P's Composite Common Stock: Dividend Yield	2
123	S&P PE ratio	S&P's Composite Common Stock: Price-Earnings Ratio	5

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